THE LEMPERT FUNCTION OF THE SYMMETRIZED POLYDISC IN HIGHER DIMENSIONS IS NOT A DISTANCE

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Abstract. We prove that the Lempert function of the symmetrized polydisc in dimension greater than two is not a distance.

1. Introduction

A consequence of the fundamental Lempert theorem (see [10]) is the fact that the Carathéodory distance and the Lempert function coincide on any domain $D \subset \mathbb{C}^n$ with the following property $(*)$ (cf. [7]):
$(*)$ *D can be exhausted by domains biholomorphic to convex domains.*

For more than 20 years it was an open question whether the converse of the above result is true in the reasonable class of domains (e.g. in the class of bounded pseudoconvex domains). In other words, does the equality between the Carathéodory distance and the Lempert function of a bounded pseudoconvex domain D imply that D has the property $(*)$.

The only counterexample so far, the so-called symmetrized bidisc $\mathbb{G}_2$, was recently discovered and discussed in a series of papers (see [2], [3], [1] and [5], see also [7]).

What remained open is the following natural question (see [7]):

Do Carathéodory distance and Lempert function coincide on the symmetrized polydisc $\mathbb{G}_n$ for any dimension $n \geq 3$?

The aim of the present paper is to give a negative answer to the above question proving that the Lempert function of $\mathbb{G}_n$ ($n \geq 3$) is not

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a distance. This implies that $\mathbb{G}_n$ ($n \geq 3$) does not have property $(*)$ (for a direct proof of this fact see [11]).

Moreover, we show that for any dimension greater than two there are bounded pseudoconvex domains not satisfying $(*)$ and for which the Carathéodory distance and the Lempert function are equal.

2. Background and results

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$. Let $\sigma_n = (\sigma_{n,1}, \dots, \sigma_{n,n}) : \mathbb{C}^n \to \mathbb{C}^n$ be defined as follows:

$$\sigma_{n,k}(z_1, \dots, z_n) = \sum_{1 \leq j_1 < \dots < j_k \leq n} z_{j_1} \dots z_{j_k}, \quad 1 \leq k \leq n.$$

The domain $\mathbb{G}_n = \sigma_n(\mathbb{D}^n)$ is called *the symmetrized n-disc.*

Recall now the definitions of the Carathéodory pseudodistance, the Carathéodory-Reiffen pseudometric, the Lempert function and the Kobayashi-Royden pseudometric of a domain $D \subset \mathbb{C}^n$ (cf. [7]):

$$c_D(z, w) = \sup\{\tanh^{-1} |f(w)| : f \in \mathcal{O}(D, \mathbb{D}), f(z) = 0\},$$
$$\gamma_D(z; X) = \sup\{|f'(z)X| : f \in \mathcal{O}(D, \mathbb{D}), f(z) = 0\},$$
$$\tilde{k}_D(z, w) = \inf\{\tanh^{-1} |\alpha| : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) : \varphi(0) = z, \varphi(\alpha) = w\},$$
$$\kappa_D(z; X) = \inf\{\alpha \geq 0 : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) : \varphi(0) = z, \alpha\varphi'(0) = X\},$$

where $z, w \in D$, $X \in \mathbb{C}^n$. The Kobayashi pseudodistance k_D (respectively, the Kobayashi–Buseman pseudometric $\hat{\kappa}_D$) is the largest pseudodistance (respectively, pseudonorm) which does not exceed $\tilde{k}_D$ (respectively, κ_D).

It is well-know that $c_D \leq k_D \leq \tilde{k}_D$, $\gamma_D \leq \hat{\kappa}_D \leq \kappa_D$, and

$$\gamma_D(z; X) = \lim_{\mathbb{C}_* \ni t \to 0} \frac{c_D(z, z + tX)}{t} \text{ (cf. [7]),}$$

and if D is taut, then

$$\kappa_D(z; X) = \lim_{\mathbb{C}_* \ni t \to 0} \frac{\tilde{k}_D(z, z + tX)}{t} \text{ (see [12]).}$$

Repeat that for $m \in \mathbb{N}$,

$$k_D^{(m)}(z, w) := \inf\{\sum_{j=1}^{m} \tilde{k}_D(z_{j-1}, z_j) : z = z_0, z_1, \dots, z_{m-1}, z_m = w \in D\}.$$

Note that $\tilde{k}_D = k_D^{(1)} \geq k_D^{(2)} \geq \dots$, $k_D = \lim_{m \to \infty} k_D^{(m)}$, and, if D is taut, then

$$\hat{\kappa}_D(z; X) = \lim_{\mathbb{C}_* \ni t \to 0} \frac{k_D(z, z + tX)}{t} \text{ (see [8]).} \tag{1}$$

For $m \in \mathbb{N}$, consider the infinitesimal version of $k_D^{(m)}$, namely

$$\kappa_D^{(m)}(z;X) = \inf\left\{\sum_{j=1}^m \kappa_D(z;X_j) : \sum_{j=1}^m X_j = X\right\}.$$

Then

$$\kappa_D = \kappa_D^{(1)} \geq \kappa_D^{(2)} \geq \dots \geq \kappa_D^{(2n-1)} \geq \kappa_D^{(2n)} = \hat{\kappa}_D$$

(for last equality see [9]). We also point out that obvious modifications in the proof of (1) in [8] show that if D is taut, then

$$\lim_{u,v\to z,\ u\neq v} \frac{k_D^{(m)}(u,v) - \kappa_D^{(m)}(z;u-v)}{||u-v||} = 0$$

uniformly in m and locally uniformly in z; thus,

$$\kappa_D^{(m)}(z;X) = \lim_{\mathbb{C}_*\ni t\to 0} \frac{k_D^{(m)}(z,z+tX)}{t}$$

uniformly in m and locally uniformly in z and X.

Note that $\mathbb{G}_n$ is a hyperconvex domain (see [6]) and, therefore, a taut domain. (Thus, all the introduced invariant functions are continuous (in both variables) for $D = \mathbb{G}_n$.) Even more, $\mathbb{G}_n$ is $c_{\mathbb{G}_n}$-finitely compact (see Corollary 3.2 in [4]).

In the proof of our main result (Theorem 1) we shall need some mappings defined on $\mathbb{G}_n$.

For $\lambda \in \overline{\mathbb{D}}$, $n \geq 2$ one may define the rational mapping

$$p_{n,\lambda} : \mathbb{C}^n \ni z = (z_1,\dots,z_n) \mapsto (\tilde{z}_1(\lambda),\dots,\tilde{z}_{n-1}(\lambda)) = \tilde{z}(\lambda) \in \overline{\mathbb{C}}^{n-1},$$

where $\tilde{z}_j(\lambda) = \dfrac{(n-j)z_j + \lambda(j+1)z_{j+1}}{n+\lambda z_1}$, $1 \leq j \leq n-1$. Then $z \in \mathbb{G}_n$ if and only if $\tilde{z}(\lambda) \in \mathbb{G}_{n-1}$ for any $\lambda \in \overline{\mathbb{D}}$ (see Corollary 3.4 in [4]).

We may also define for $\lambda_1,\dots,\lambda_{n-1} \in \overline{\mathbb{D}}$ the rational function

$$f_{\lambda_1,\dots,\lambda_{n-1}} = p_{2,\lambda_1} \circ \dots \circ p_{n,\lambda_{n-1}} : \mathbb{C}^n \mapsto \overline{\mathbb{C}}.$$

Observe that

$$f_\lambda(z) := f_{\lambda,\dots,\lambda}(z) = \frac{\sum_{j=1}^n jz_j\lambda^{j-1}}{n + \sum_{j=1}^{n-1}(n-j)z_j\lambda^j}.$$

By Theorem 3.2 in [4], $z \in \mathbb{G}_n$ if and only if $\sup\limits_{\lambda\in\overline{\mathbb{D}}} |f_\lambda(z)| < 1$. In fact, by Theorem 3.5 in [4], if $z \in \mathbb{G}_n$, then the last supremum is equal to $\sup\limits_{\lambda_1,\dots,\lambda_{n-1}\in\overline{\mathbb{D}}} |f_{\lambda_1,\dots,\lambda_{n-1}}(z)|$.

It follows that

$$c_{\mathbb{G}_n}(z,w) \geq p_{\mathbb{G}_n}(z,w) := \max_{\lambda_1,\dots,\lambda_{n-1}\in\mathbb{T}} |p_{\mathbb{D}}(f_{\lambda_1,\dots,\lambda_{n-1}}(z), f_{\lambda_1,\dots,\lambda_{n-1}}(w))|,$$

where $\mathbb{T} = \partial\mathbb{D}$ and $p_{\mathbb{D}}$ is the Poincaré distance; in particular,

$$\gamma_{\mathbb{G}_n}(0;X) \geq \lim_{\mathbb{C}_*\ni t\to 0} \frac{p_{\mathbb{G}_n}(0,tX)}{|t|} = \max_{\lambda\in\mathbb{T}} |\tilde{f}_\lambda(X)| =: \rho_n(X),$$

where

$$\tilde{f}_\lambda(X) = \frac{\sum_{j=1}^n jX_j\lambda^{j-1}}{n}.$$

Let $e_1,\dots,e_n$ be the standard basis of $\mathbb{C}^n$ and $L_{k,l} = \{X \in \mathbb{C}^n : X = X_ke_k + X_le_l\}$, $1 \leq k \leq l \leq n$. Observe that if $X \in L_{k,l}$, then

$$\rho_n(X) = \frac{k|X_k| + l|X_l|}{n}.$$

For $n = 2$ one has that $\kappa_{\mathbb{G}_2} \equiv c_{\mathbb{G}_2} \equiv p_{\mathbb{G}_2}$ (see [1, 2]). On the other hand, we have the following.

Theorem 1. *Let $n \geq 3$.*

(a) If k divides n, then $\kappa_{\mathbb{G}_n}(0;e_k) = \rho_n(e_k)$. Therefore, if l also divides n, then $\kappa^{(2)}_{\mathbb{G}_n}(0;X) = \rho_n(X)$ for any $X \in L_{k,l}$.

(b) If k does not divide n, then $\hat{\kappa}_{\mathbb{G}_n}(0;e_k) > \rho_n(e_k)$.

(c) If $X \in L_{1,n} \setminus (L_{1,1} \cup L_{n,n})$, then $\kappa_{\mathbb{G}_n}(0;X) > \rho_n(X)$.

In particular, $k_{\mathbb{G}_n}(0,\cdot) \not\equiv p_{\mathbb{G}_n}(0,\cdot)$, $\tilde{k}_{\mathbb{G}_n}(0,\cdot) \not\equiv k^{(2)}_{\mathbb{G}_n}(0,\cdot)$, and $\mathbb{G}_n$ does not have property $(*)$.

Remarks. (i) We already know that for $n \geq 3$ at least one of the identities $\hat{\kappa}_{\mathbb{G}_n}(0,\cdot) \equiv \gamma_{\mathbb{G}_n}(0,\cdot)$ and $\gamma_{\mathbb{G}_n}(0,\cdot) \equiv \rho_n$ does not hold and, therefore, the same applies to the identities $k_{\mathbb{G}_n}(0,\cdot) \equiv c_{\mathbb{G}_n}(0,\cdot)$ and $c_{\mathbb{G}_n}(0,\cdot) \equiv p_{\mathbb{G}_n}(0,\cdot)$. It will be interesting to know if however some of them hold and whether $c^i_{\mathbb{G}_n}(0,\cdot) \equiv c_{\mathbb{G}_n}(0,\cdot)$ ($c^i_{\mathbb{G}_n}$ denotes the inner Carathéodory distance of $\mathbb{G}_n$).

(ii) Observe that $\mathbb{G}_{2n|_{L_{n,2n}}} = \mathbb{G}_2$. Then, in contrast to (c), for $z,w \in L_{n,2n}$ one has that

$$p_{\mathbb{G}_{2n}}(z,w) \leq \tilde{k}_{\mathbb{G}_{2n}}(z,w) \leq \tilde{k}_{\mathbb{G}_2}(z,w) = p_{\mathbb{G}_2}(z,w) \leq p_{\mathbb{G}_{2n}}(z,w)$$

and therefore $\tilde{k}_{\mathbb{G}_{2n}}(z,w) = p_{\mathbb{G}_{2n}}(z,w)$.

In spite of Theorem 1, for any $n \geq 3$ there are bounded pseudoconvex domains $D \subset \mathbb{C}^n$ which do not have the property $(*)$ but nevertheless $c_D \equiv \tilde{k}_D$.

Theorem 2. *Let $G \subset \mathbb{C}^m$ be a balanced domain (that is, $\lambda z \in G$ for any $\lambda \in \overline{\mathbb{D}}$ and any $z \in G$). Then $D = \mathbb{G}_2 \times G$ does not fulfill the property $(*)$.*

On the other hand, if, in addition, G is convex (for example, G is the unit polydisc or the unit ball), then $c_D \equiv \tilde{k}_D$.

3. Proofs

Proof of Theorem 1. (a) We shall prove even more, namely, that $\kappa_{\mathbb{G}_n}(0; e_k) = \rho_n(e_k)$ if and only if k divides n.

Assume that $\kappa_{\mathbb{G}_n}(0; e_k) = \rho_n(e_k)$. Since $\mathbb{G}_n$ is a taut domain, there exists an extremal mapping for $\kappa_{\mathbb{G}_n}(0; e_k)$, that is, a holomorphic mapping $\varphi : \mathbb{D} \to \mathbb{G}_n$ with $\varphi(z) = (z\varphi_1(z), \dots, z\varphi_n(z))$, $\varphi_k(0) = 1/\rho_n(e_k) = n/k$ and $\varphi_j(e_j) = 0$ for $1 \le j \le n$, $j \ne k$. Observe that $f_\lambda \circ \varphi \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ and $f_\lambda \circ \varphi(0) = 0$ for any $\lambda \in \overline{\mathbb{D}}$. It follows by the maximum principle that $g_\lambda \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$, where

$$g_\lambda(z) = \frac{\sum_{j=1}^n j\varphi_j(z)\lambda^{j-1}}{n + \sum_{j=1}^{n-1}(n-j)z\varphi_j(z)\lambda^j}.$$

Since $g_\lambda(0) = \lambda^{k-1}$, the maximum principle implies that $g_\lambda \equiv \lambda^{k-1}$, that is

$$\sum_{j=1}^n j\varphi_j(z)\lambda^{j-1} = n\lambda^{k-1} + z\sum_{j=1}^{n-1}(n-j)\varphi_j(z)\lambda^{k+j-1}, \qquad \lambda \in \overline{\mathbb{D}}, z \in \mathbb{D}.$$

Comparing the respective coefficients of these two polynomials of λ, we get that $\varphi_k \equiv \dfrac{n}{k}$, $\varphi_1 \equiv \dots \equiv \varphi_{k-1} \equiv \varphi_{n+1-k} \equiv \dots \equiv \varphi_{n-1} \equiv 0$ and

$$(k+j)\varphi_{k+j}(z) \equiv (n-j)z\varphi_j(z),\ 1 \le j \le n-k.$$

These relations imply that $\varphi_j \equiv 0$ if k does not divide j, and $\varphi_j \equiv \binom{n/k}{j/k} z^{j/k-1}$ if k divides j. If k does not divide n, then $n-k < k[n/k] < n$ and hence $\varphi_{k[n/k]} \equiv 0$, a contradiction. Conversely, if k divides n, then put $\varphi = (\tilde{\varphi}_1, \dots, \tilde{\varphi}_n)$, where $\tilde{\varphi}_j \equiv 0$ if k does not divide j and $\tilde{\varphi}_j(z) = \binom{n/k}{j/k} z^{j/k}$ if k divides j. It follows from the proof above that φ sends $\mathbb{D}$ into $\mathbb{G}_n$, and, up to a rotation, it is the only extremal mapping for $\kappa_{\mathbb{G}_n}(0; e_k) = \rho(e_k)$.

To see that if k and l divide n, then $\kappa^{(2)}_{\mathbb{G}_n}(0; X) = \rho_n(X)$ for any $X \in L_{k,l}$, it is enough to observe that

$$\rho_n(X) \le \kappa^{(2)}_{\mathbb{G}_n}(0; X) \le \kappa_{\mathbb{G}_n}(0; X_k e_k) + \kappa_{\mathbb{G}_n}(0; X_l e_l)$$

$$= \rho_n(X_k e_k) + \rho_n(X_l e_l) = \rho_n(X).$$

(b) Denote by I, J, and K the indicatrices of ρ_n, $\hat{\kappa}_{\mathbb{G}_n}(0;\cdot)$, and $\kappa_{\mathbb{G}_n}(0;\cdot)$, respectively ($I = \{X \in \mathbb{C}^n : \rho_n(X) < 1\}$ and etc.). Note that if $X \in \overline{J}$ is an extreme point of $\overline{I}$, then X is an extreme point of $\overline{J}$ and therefore $X \in \overline{K}$. Thus, (b) follows by the inequality $\kappa_{\mathbb{G}_n}(0; ne_k/k) > 1$ and the fact that ne_k/k is an extreme point of $\overline{I}$. In fact, to see the last claim observe that if $0 < \alpha < 1$, $\rho_n(X) = \rho_n(Y) = 1$, $ne_k/k = \alpha X + (1-\alpha)Y$ and $\lambda \in \mathbb{T}$, then

$$\begin{aligned} 1 = \lambda^{1-k}\tilde{f}_\lambda(ne_k/k) &\le \alpha|\tilde{f}_\lambda(X)| + (1-\alpha)|\tilde{f}_\lambda(Y)| \\ &\le \alpha\rho_n(X) + (1-\alpha)\rho_n(Y) = 1. \end{aligned}$$

Hence, $\tilde{f}_\lambda(X) = \tilde{f}_\lambda(Y) = \lambda^{k-1}$ for any $\lambda \in \mathbb{T}$, that is, $X = Y = ne_k/k$.

(c) First, note that if $\lambda \in \mathbb{T}$, then the mapping $(z_1, z_2, \dots, z_n) \to (\lambda z_1, \lambda^2 z_2, \dots, \lambda^n z_n)$ is an automorphism of $\mathbb{G}_n$ and

$$\kappa_{\mathbb{G}_n}(0; \lambda X) = \kappa_{\mathbb{G}_n}(0; X).$$

Applying these facts, we may assume that $X_1, X_n > 0$.

Since

$$\begin{aligned} \kappa_{\mathbb{G}_n}(0;X) &\ge \kappa_{\mathbb{G}_{n-1}}(p_{n,1}(0); p'_{n,1}(0)(X)) \\ &= \kappa_{\mathbb{G}_{n-1}}\left(0; \frac{n-1}{n} X_1 e_1 + X_n e_{n-1}\right), \end{aligned}$$

it follows by induction on n that $\kappa_{\mathbb{G}_n}(0;X) \ge \kappa_{\mathbb{G}_3}(0;Y)$, where $Y = \dfrac{3X_1}{n} e_1 + X_n e_3$. Assume that $\kappa_{\mathbb{G}_n}(0;X) = \rho_n(X)$. Then

$$\rho_n(X) \ge \kappa_{\mathbb{G}_3}(0;Y) \ge \rho_3(Y) = \rho_n(X)$$

and hence $\kappa_{\mathbb{G}_3}(0;Y) = \rho_3(Y)$. Now, taking an extremal mapping $\varphi(z) = (z\varphi_1(z), z\varphi_2(z), z\varphi_n(z))$ for $\kappa_{\mathbb{G}_3}(0;Y)$, with the same notations as in the proof of (a), we obtain that $g_\lambda \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$, $\lambda \in \mathbb{T}$. Since $g_{\pm 1}(0) = 1$, then $g_{\pm 1} \equiv 1$, that is

$$\varphi_1(z) \pm 2\varphi_2(z) + 3\varphi_3(z) = 3 \pm 2z\varphi_1(z) + z\varphi_2(z).$$

Thus,

$$\varphi_2(z) \equiv z\varphi_1(z) \text{ and } \varphi_3(z) \equiv 1 + \frac{z^2-1}{3}\varphi_1(z).$$

Set $\psi = \varphi_1(z)/3$. Then $g_\lambda \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$ means that

$$\begin{aligned} &\left|\frac{\psi(z) + 2\lambda z\psi(z) + \lambda^2(1 + (z^2-1)\psi(z))}{1 + 2\lambda z\psi(z) + \lambda^2 z^2\psi(z)}\right| \le 1 \\ \iff &\left|\frac{\psi(z)(1+\lambda z)^2 + \lambda^2(1-\psi(z))}{\psi(z)(1+\lambda z)^2 + 1 - \psi(z)}\right| \le 1 \end{aligned}$$

$$\iff \operatorname{Re}(\psi(z)(1-\overline{\psi(z)})((\overline{\lambda}+z)^2-(1+\lambda z)^2)) \leq 0.$$

If $\lambda = x+iy, z = iw, w \in \mathbb{R}, a = \operatorname{Re}(\psi(z)) - |\psi(z)|^2, b = \operatorname{Im}(\psi(z))$, then

$$y(a(2w-y(w^2+1))+bx(1-w^2)) \leq 0,\ \forall\ x^2+y^2=1.$$

Setting $x=0$ implies that $a \geq 0$. Letting $y \to 0^+$ gives $-2aw \geq (1-w^2)|b|$. Hence $a=b=0$ if $w>0$. Then the identity principle implies that either $\psi \equiv 0$ or $\psi \equiv 1$. Thus, either $X_1 = 0$ or $X_n = 0$ which a contradiction. □

Proof of Theorem 2. The second part follows by the equalities $c_{\mathbb{G}_2} = \tilde{k}_{\mathbb{G}_2}$ and $c_G = \tilde{k}_G$, and the product property of c_D and $\tilde{k}_D$: $c_D = \max\{c_{\mathbb{G}_2}, c_G\}$ and $\tilde{k}_D = \max\{\tilde{k}_{\mathbb{G}_2}, \tilde{k}_G\}$ (cf. [7]).

The proof of the first part the proof in [5] that $\mathbb{G}_2$ does not have property $(*)$. For convenience of the reader, we include it.

Let $h_1(z_1+z_2, z_1z_2) = \max\{|z_1|, |z_2|\}$, $h_2(z) = \inf\{t>0 : z/t \in G\}$ (the Minkowski function of G), $h = \max\{h_1, h_2\}$ and

$$\pi_\lambda(z_1,\dots,z_{m+2}) = (\lambda z_1, \lambda^2 z_2, \lambda z_3, \dots, \lambda z_{m+2}),\ \lambda \in \mathbb{C}.$$

Note that $h(\pi_\lambda(z) = |\lambda| h(z)$ and $D = \{z \in \mathbb{C}^{m+2} : h(z) < 1\}$.

Assume now that D fulfills the property $(*)$. Take two points $a, b \in \mathbb{G}_2 \times \{0\} \subset D$. We may find an $\varepsilon > 0$ and a domain $D_\varepsilon \subset \{h < 1-\varepsilon\}$ which is biholomorphic to a convex domain $\tilde{D}_\varepsilon$ and such that $\lambda a, \lambda b \in D_\varepsilon$ for $\lambda \in \overline{\mathbb{D}}$. Let $\varphi_\varepsilon : D_\varepsilon \to \tilde{D}_\varepsilon$ be the corresponding biholomorphic mapping. We may assume that $\varphi_\varepsilon(0) = 0$ and $\varphi_\varepsilon'(0) = \mathrm{id}$. Note that

$$g_\varepsilon(\lambda) = \varphi_\varepsilon^{-1}\left(\frac{\varphi_\varepsilon(\pi_\lambda(a)) + \varphi_\varepsilon(\pi_\lambda(b))}{2}\right),$$

is a holomorphic mapping from a neighborhood of $\overline{\mathbb{D}}$ into D. We have

$$g_\varepsilon(0) = 0, g'_{\varepsilon,1}(0) = \frac{a_1+b_1}{2}, g'_{\varepsilon,2}(0) = 0, \dots, g'_{\varepsilon,m+2}(0) = 0$$

$$\text{and } g'_{\varepsilon,2}(0) = a_2 + b_2 + \frac{c_\varepsilon}{4}(a_1-b_1)^2, \text{ where } c_\varepsilon = \frac{\partial^2 \varphi_{\varepsilon,2}}{\partial z_1^2}(0).$$

Thus, the mapping $f_\varepsilon(\lambda) = \pi_{1/\lambda} \circ g_\varepsilon(\lambda)$ can be extended at 0 as

$$f_\varepsilon(0) = \left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2} + \frac{c_\varepsilon}{8}(a_1-b_1)^2, 0, \dots, 0\right).$$

Since D is assumed to satisfy the property $(*)$, it should be pseudoconvex, that is, h is a plurisubharmonic function. Then the maximum principle implies that $h(f_\varepsilon(0)) \leq \max_{|\lambda|=1} h(f_\varepsilon(\lambda)) < 1$ which means that $f_\varepsilon(0) \in D$.

Assuming now that $\lim_{\varepsilon\to 0^+} c_\varepsilon \neq 0$ and having in mind that c_ε is bounded, we may find $c \neq 0$ such that

$$m = \left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2} + c(a_1-b_1)^2, 0, \dots, 0\right) \in \overline{D}$$

for any $a, b \in \overline{\mathbb{G}_2} \times \{0\}$. Taking $\alpha = e^{i(\arg(c)+\pi)/2}$, $a_1 = \alpha + 1$, $a_2 = \alpha$, $b_1 = \alpha - 1$, $b_2 = -\alpha$, we obtain that $1 \geq h_1(m) = \dfrac{1+\sqrt{1+16c}}{2}$ which is impossible.

Thus, $\lim_{\varepsilon\to 0^+} f_\varepsilon(0) = \dfrac{a+b}{2} \in \overline{D}$ for any $a, b \in \mathbb{G}_2 \times \{0\}$, that is, $\mathbb{G}_2$ is a convex domain, a contradiction (for example, $(2,1), (2i,-1) \in \partial\mathbb{G}_2$, but $(1+i, 0) \notin \overline{\mathbb{G}_2}$).

$\square$

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